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Asymptotic entanglement dynamics and geometry of quantum states

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Abstract
A given dynamics for a composite quantum system can exhibit several distinct properties for the asymptotic entanglement behavior, such as entanglement sudden death, asymptotic death of entanglement, sudden birth of entanglement, etc. A classification of the possible situations was given by Terra Cunha (2007 *New J. Phys.* **9** 237) but for some classes there were no known examples. In this work, we give a better classification for the possible relaxing dynamics in light of the geometry of their set of asymptotic states and give explicit examples for all the classes. Although the classification is completely general, in the search for examples it is sufficient to use two qubits with dynamics given by differential equations in the Lindblad form (some of them are non-autonomous). We also investigate, in each case, the probabilities of finding each possible behavior for random initial states.

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1. Introduction

Entanglement is a fundamental property of composite quantum systems, first noted by Schrödinger [1]. The best knowledge of the whole of a composite quantum system may not include complete knowledge of its parts. It has strong conceptual implications on physics, since it is a property that has no classical analogue, so we are forced to change significantly our perspective of nature. Such a peculiar character allows it to be considered as a fundamental resource for some non-classical tasks such as teleportation of a quantum state [2], quantum computation [3], quantum cryptography [4], etc[3]. Once entanglement is considered a resource,

[3] Entanglement is not necessary for quantum key distribution; however it is used in the best-known proof of security of such protocols [5].

it seems natural to quantify it [6]. In all the applications named above, it is necessary to optimize the amount of entanglement in a suitable composite quantum system to best execute the desired task.

Real quantum systems always interact with their environment, irrespective of the efforts to protect it. This interaction will, in general, create some entanglement between the quantum system and the environment, and this entanglement will, somewhat ironically, spoil the entanglement between the parts of the 'useful' system (for bipartite systems, this affirmation has a precise meaning provided by the monogamy of the entanglement theorem [7]).

While in most of the models used to describe quantum open systems the coherences of a state decay asymptotically to zero, it was recently recognized that entanglement may 'die' at finite time [8], a phenomenon called entanglement sudden death [9]. This phenomenon has attracted some attention, specially connected to the difficulty of keeping entanglement alive for its uses as a resource. Some interesting generalizations were studied [10] and some experiments were proposed [11] and realized [12]. This phenomenon, though, has a simple explanation if one looks at the geometry of quantum states [13]. Namely, while the set of 'decohered' states always has zero volume inside the set of all possible quantum states, the set of separable states has not only a positive volume but also a non-empty interior [14] when the global system has a finite dimensional Hilbert space.

The geometrical approach to the problem allows one to classify the dynamics of a quantum system according to the geometry of its asymptotic states (if the dynamics implies them) relative to the set of separable states [13]. In the cited paper some classes were exemplified, but at that time it was not clear whether all *a priori* possible situations could be found.

In this paper, we review the geometric classification of entanglement dynamics and provide explicit examples to all *a priori* possible situations. All examples are given in the context of two-qubit Lindblad differential equations, with some cases using non-autonomous equations (exactly those in the classes for which examples were not previously known). We also introduce a new analysis of how often each specific behavior occurs for a given dynamics, in the light of probability theory applied to the set of initial states [15].

2. The geometry of entanglement sudden death: general picture

What can we say about the geometry of entanglement or the geometry of the set of separable states, for general multipartite systems? First of all, it can be said that the set of separable states is closed, convex and with a non-empty interior (we shall assume finite dimensional Hilbert spaces throughout the paper). Its complement relative to the set of quantum states also has a non-empty interior and is certainly non-convex. Actually, in general, its complement is much larger, i.e. it has greater volume (if one considers the Hilbert–Schmidt metric, for instance). An extremely oversimplified illustration of this situation is given in figure 1. Here, we denote by D the set of all quantum states and we consider it as immersed in the set $\mathcal{A}$ of Hermitian matrices of unity trace, S the subset composed of the separable states, ∂S and ∂D their boundaries relative to D and $\mathcal{A}$, respectively, and $E = D - S$ the set of entangled states. The boundary of the set of quantum states is composed of all those states which have at least one zero eigenvalue, so, in particular, it contains all the pure states. Note that there are both entangled and separable pure states in ∂D. Actually, more than that, the 'area' of the separable states inside ∂D is non-zero [16].

Let us consider a dynamics with a non-trivial stationary set, St. By a stationary set, we mean that for every initial state ρ and open set $V \supseteq St$ we have that $\rho(t)$ (the state at time t) belongs to V for all sufficiently large t. Of course, if some dynamics accepts a set of

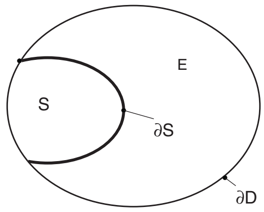

Figure 1. Diagram of the set of entangled states.

stationary states, this set will be the smallest stationary set of the dynamics. Anyway, from the simple picture given in figure 1, and considering the location of St in it, we may distinguish three possibilities which have consequences on the asymptotic dynamics of entanglement: (i) $St \subset \mathrm{Int}(S)$ implies that every initially entangled state will lose all of its entanglement at finite time (sudden death of entanglement); (ii) if $St \cap \partial S \neq \emptyset$, then based on only this information, many situations can occur: asymptotic or sudden death of entanglement and non-zero asymptotic entanglement; (iii) if $St \subset E$, every initial state exhibits some entanglement asymptotically.

The complete classification must yet consider that the stationary set, St, can consist of a single state (e.g., thermal equilibrium state) or of a non-trivial set (e.g., for phase reservoirs). In this sense, each situation above gives rise to two cases, in a total of six classes.

Note that, in cases (ii) and (iii), if we start with a separable state it is possible in the first one and certain in the second that entanglement will be created, a situation which may be called sudden birth of entanglement [13]. It is good to stress that since the only information we have about the dynamics is some partial information about a stationary set, anything may happen with the entanglement for short times: it may die, resurrect, oscillate, etc. It is also important to mention that such analysis does not depend on the specific entanglement quantifier used to follow the dynamics, but only on the assumption that it is continuous and strictly positive on entangled states.

Given a dynamics that fits in case (ii), one can in general find examples of initial states whose entanglement die asymptotically or suddenly [11]. An interesting way to have a global view of the properties of this dynamics in this respect is through the following question: if one picks a random initial state, what is the most probable situation, asymptotic or sudden death? That is, if the dynamics can exhibit both of these properties, what is the most typical one? To answer this question, one must formulate it properly. Fixing a dynamics for a composite system with state space D with a suitable probability measure P on it and a continuous entanglement quantifier $e : D \to \mathbb{R}_+$, with $e(E) \subset (0, \infty)$, we define the following events (subsets of D, in the language of probability theory) whose probabilities may be of interest:

- states that exhibit sudden death of entanglement: $SDE = \{\rho \in \mathcal{D} | \exists t_0, t_1$ such that $E(\rho(t_0)) > 0$ and $E(\rho(t)) = 0$ for all $t > t_1\}$,
- states that exhibit asymptotic death of entanglement: $ADE = \{\rho \in \mathcal{D} | \exists (t_n)_{n=1}^{\infty}, t_n \to \infty$, such that $E(\rho(t_n)) > 0$ and $\lim_{t\to\infty} E(\rho(t)) = 0\}$,

where $\rho(t)$ denotes the time t evolution of initial state ρ according to the dynamics. Note that these definitions do not coincide strictly with the common sense of such notions since in general one only looks for initial states that already have some entanglement, which is not necessary here: an initially separable state can, in principle, acquire some entanglement that will subsequently die (suddenly or asymptotically). The strict notion would be given by the events:

- $SDE' = SDE \cap E$,
- $ADE' = ADE \cap E$.

If the asymptotic dynamics exhibit entangled states, one can also look at the following events.

- The states exhibit entanglement asymptotically: $AE = \{\rho \in \mathcal{D} | \exists t_0, c > 0$ where $E(\rho(t)) > c$ for all $t > t_0\}$.
- An initially separable state acquires entanglement asymptotically (sudden birth of entanglement): $SBE = \{\rho \in \mathcal{D} | E(\rho) = 0$ and $\exists t_0, c > 0$ where $E(\rho(t)) > c$ for all $t > t_0\}$

(note that $SBE = AE \cap S$).

Instead of choosing a specific probability measure to deal with, our results will only require that it is non-singular, i.e. sets contained in sub-manifolds of D with dimensions strictly smaller than the dimension of D have zero probability. The problem of computing the probability (or volume) of the event (set) S exactly is still an open issue for the most natural probability measures. Several bounds and estimates exist for several probability measures and events [16, 17].

3. Explicit examples

Given the general picture we may now look at some concrete examples where most of them, as we will see, are very natural and experimentally feasible. The simplest type of dynamics, namely one that is convex-linear, Markovian and completely positive, will suffice to provide rich examples. It will be sufficient to work with the simplest composite system, two qubits, in order to exhibit examples for all classes of dynamics. Considering that we deal with a system with a finite Hilbert space, we may apply the Lindblad theorem and describe the map by an ordinary, linear, first-order differential equation with the form given by [18]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \mathcal{D}[\rho], \tag{1a}$$

where

$$\mathcal{D}[\rho] = \sum_j \gamma_j \big(2A_j \rho A_j^\dagger - A_j^\dagger A_j \rho - \rho A_j^\dagger A_j\big), \tag{1b}$$

γ_j are real constant numbers, A_j are general linear operators and H is a Hermitian operator. This type of dynamics has the advantage that it is simple to find its asymptotic states; in general, one just has to look at the kernel of the 'superoperator' $\mathcal{L}$, which is a linear operator that can be understood to be defined over the set of 4×4 complex matrices or the subset of Hermitian matrices, a real vector space, and take its intersection with the set of mixed states. It is curious to note that in order to find the two missed examples in [13], we had to allow for non-autonomous Lindblad equations, that is, equations with the same form but with parameters γ_j varying in time. For this type of dynamics, the set of stationary states does not need to be the intersection of a subspace with the set of quantum states.

3.1. Localizing a two-qubit state in the set of all states

The set of all quantum states for a composite system can be divided geometrically according to the dichotomy $\{\mathrm{Int}\, D, \partial D\}$ and the trichotomy $\{\mathrm{Int}\, S, \partial S, E\}$. Dealing with the special case of two qubits has the advantage that one can easily infer the location of a state according to this subdivision with the help of $\mathrm{Det}\,\rho$ and $\mathrm{Det}\,\rho^\Gamma$, the determinants of the state and of its partial transpose respectively. Both of these functions are continuous in all natural metrics, Hilbert–Schmidt, etc, i.e. we know that small perturbations of a state in a given metric imply small perturbations of the values of both quantities. So if, e.g., both of them are positive for a given state, we can find a neighborhood of that state where these quantities remain with the same sign. Then, the determinant of the operator tells us if it is in the interior or in the border of D (if it is greater than or equal to zero, respectively). The determinant of the partial transpose, on the other hand, gives us complete information about its entanglement because it is known [19] that the state is entangled iff the determinant is strictly negative. Thus, this determinant tells us if the state is in the interior of the set of separable states (if it is grater than zero), in the border ∂S (if it is equal to zero) or inside the set of the entangled states (if it is strictly negative). Finally, if ρ is a given state with $d = \mathrm{Det}\,\rho$ and $d^\Gamma = \mathrm{Det}\,\rho^\Gamma$, we can have the following.

(i) $d > 0$ and $d^\Gamma > 0$: the state is in the interior of D and in the interior of S relative to D, i.e. it belongs to $S - \partial S$ (e.g., the completely mixed state, $\rho_{\mathrm{mix}} = I/4$).
(ii) $d > 0$ and $d^\Gamma = 0$: the state is in the interior of D and in ∂S (e.g., the state $\frac{2}{3}\rho_{\mathrm{mix}} + \frac{1}{3}\rho_{\mathrm{singlet}}$, where ρ_{singlet} refers to the state in equation (2) with $a = 0$, $b = -c = 1/2$).
(iii) $d > 0$ and $d^\Gamma < 0$: the state is in the interior of D and belongs to E (e.g., the Werner states [20] $p\rho_{\mathrm{mix}} + (1 - p)\rho_{\mathrm{singlet}}$, for $0 < p < 2/3$).
(iv) $d = 0$ and $d^\Gamma > 0$: the state is in the border of D and in $S - \partial S$ (recording that we defined ∂S as the boundary relative to D, while ∂D is relative to $\mathcal{A}$). For instance, if $a > b > 0$, $2a + 2b = 1$ and $|c| = b$:

$$\rho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}. \tag{2}$$

(v) $d = 0$ and $d^\Gamma = 0$: the state is in $\partial D \cap \partial S$ (e.g., a separable pure state).
(vi) $d = 0$ and $d^\Gamma < 0$: the state is in the border of D and belongs to E (e.g., ρ_{singlet}).

With all these tools in hand, we can arrive at the following examples.

Case 1(a). One asymptotic state in Int (S)

Perhaps the most natural example of this situation is the case where both qubits, which we shall call A and B, are spatially well-separated two-level atoms, interacting with thermal fields. The separation between them implies that the thermal reservoirs are independent. The Lindblad equation that describes this dynamics is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H_A + H_B, \rho] + \mathcal{D}_A \otimes I[\rho] + I \otimes \mathcal{D}_B[\rho], \tag{3a}$$

where

$$\mathcal{D}_i[\rho] = \gamma_i(2\sigma_{+,i}\rho\sigma_{-,i} - \sigma_{-,i}\sigma_{+,i}\rho - \rho\sigma_{-,i}\sigma_{+,i}) + \gamma_i'(2\sigma_{-,i}\rho\sigma_{+,i} - \sigma_{+,i}\sigma_{-,i}\rho - \rho\sigma_{+,i}\sigma_{-,i}), \tag{3b}$$

with $\sigma_{\pm,i}$ being the Pauli operators for qubit i, $H_i = \frac{\hbar\omega_i}{2}\sigma_{z,i}$ the Hamiltonian for qubit i and γ_i, γ_i' non-negative constants (related to the average photon number in the field, the polarization of atoms, their coupling to the environment, etc).

It is easy to show that the system will evolve to a product state with both qubits in their respective Gibbs states, $Z_i^{-1}\,\mathrm{e}^{-\beta H_i}$, $Z_i = \mathrm{Tr}\,\mathrm{e}^{-\beta H_i}$. If the temperature is positive, the resulting state is a product state with a diagonal density matrix (in the product basis) with every diagonal entrance being non-zero. We then have that $\rho_{\mathrm{st}} = \rho_{\mathrm{st}}^\Gamma$ and $\mathrm{Det}\,\rho^\Gamma = \mathrm{Det}\,\rho > 0$. As mentioned earlier, if an initial state has some entanglement it will certainly die at finite time.

The events defined in section 2 are trivial in this case: $ADE = ADE' = SBE = AE = \emptyset$ and $SDE = SDE' = E$, so $P(SDE) = P(E)$ or $P(SDE|E) = 1$, that is, the only condition for having entanglement sudden death is that the initial state is entangled. To calculate the exact probability of this event is thus as difficult as determining the volume of the set of separable states [16].

Case 1(b). Several asymptotic states in Int S

To obtain an equation of motion for the state satisfying this propriety, namely being a relaxing dynamics with more than one asymptotic state but all of them in the interior of S, we had to appeal to a non-autonomous Lindblad equation. A dynamics that achieves the desired result would be given by a Lindblad equation with the same form as that used in the last section, describing two qubits interacting with independent reservoirs, but now with the coupling 'constants' decaying exponentially, that is, performing the correspondence $\gamma_i \mapsto \gamma_{i0}\exp(-\kappa t)$. The physical situation corresponding to the equation, although artificial, is certainly not prohibited: in principle, one can have a good control of the interaction of the qubits with their reservoir and turn it off exponentially.

To prove the result, let us write the dynamical equation in the form (in the interaction picture)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathrm{e}^{-\kappa t}\mathcal{D}[\rho(t)], \tag{4}$$

where $\mathcal{D}$ is the dissipator of the Lindbladian in the last example. For $\rho(t)$ a solution to this equation, we can define $\bar{\rho}(t) = (\rho \circ g)(t)$, where

$$g(t) = \int_0^t \mathrm{e}^{-\kappa t'}\,\mathrm{d}t'$$

is an invertible function. Substituting $\bar{\rho}$ in equation (4), we obtain an equation of motion for it:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\rho}(t) = \mathcal{D}[\bar{\rho}(t)]. \tag{5}$$

That is, $\bar{\rho}$ obeys the same dynamics of two qubits in independent thermal reservoirs with constant coupling in time, with a known solution. To find the asymptotic set for the dynamics of equation (4) is sufficient to note that since $\rho(t) = (\bar{\rho} \circ g^{-1})(t)$, $\rho(t \to \infty) = \bar{\rho}(g^{-1}(t \to \infty)) = \bar{\rho}(1/\kappa)$.

Geometrically, the autonomous dynamics given by equation (5) deforms continually the set of states D to the point ρ_{Gibbs} (i.e. provides a homotopy between them), while the time-varying version reparametrizes this deformation. The set of asymptotic states of equation (4) is then given by this deformation in the intermediate time κ^{-1}. Making κ small enough, we can assure that the asymptotic set is entirely contained in Int S, since ρ_{Gibbs} belongs to Int S, an open set.

Of course, the events SDE, ADE, etc, and their respective probabilities are exactly the same as in the last example.

We finally note that although the discussion about entanglement does not depend on whether we are dealing with the interaction or Schrödinger pictures (because the correspondence between them is given by local unitary transformations), the dynamics is not relaxing in the former. Since the state will be given by $\rho_S(t) = \exp(\mathrm{i}Ht)\rho(t)\exp(-\mathrm{i}Ht)$ and $\lim_{t\to\infty}\rho(t)$ will not, in general, commute with the exponentials, the state evolution $\rho_S(t)$ will not converge. Nevertheless, the dynamics will have an asymptotic set in the general sense discussed in section 2, namely although an initial state does not necessarily converges, one can find open sets such that the state trajectory will be confined inside them after a certain instant of time. In this particular example, one can find such open sets that are entirely contained in S.

Case 2(a). One asymptotic state in ∂S
Equation (3) also provides an example where we have only one stationary state in the border between separable and entangled states, namely the case where the qubits are subjected to two independent thermal reservoirs at *null temperature*. In this case, the stationary state is the pure state $\rho_{\rm st} = |00\rangle\langle 00|$. Again, it is diagonal in the computational basis so $\operatorname{Det} \rho_{\rm st}^{\Gamma} = \operatorname{Det} \rho_{\rm st} = 0$. Then, a neighborhood of this state always contains separable as well as entangled states. As mentioned in section 2, in this example, depending on the initial state, both behavior can happen: asymptotic and sudden death of entanglement. In fact, given an initial state with matrix elements ρ_{ij} one can shown that the determinant of the partial transpose of the state in time t will be given by

$$\operatorname{Det} \rho^{\Gamma}(t) = \mathrm{e}^{-4\kappa t} \operatorname{Det}[\rho' + \rho''(t)], \tag{6a}$$

where

$$\rho' = \begin{pmatrix} \rho_{11} & \rho_{12}^{*} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{11} + \rho_{22} & \rho_{14} & \rho_{24} + 2\rho_{13} \\ \rho_{13}^{*} & \rho_{14}^{*} & \rho_{11} + \rho_{33} & \rho_{34}^{*} + 2\rho_{12}^{*} \\ \rho_{23}^{*} & \rho_{24}^{*} + 2\rho_{13}^{*} & \rho_{34} + 2\rho_{12} & 1 \end{pmatrix}, \tag{6b}$$

and $\rho''(t)$ is a matrix which depends on ρ but where all elements decay (exponentially) to zero. Hence, as long as $\operatorname{Det} \rho' \neq 0$, the asymptotic sign of $\operatorname{Det} \rho^{\Gamma}(t)$ will be given by the sign of the determinant of ρ'. By assuming non-singular probability measure in the set of quantum states, we conclude that the event defined by the condition $\operatorname{Det} \rho' = 0$ has zero probability and can be discarded to compute the probabilities of $ADE(=ADE')$ or $SDE(=SDE')$. From the form of ρ' it is easy to find initial states such that $\operatorname{Det} \rho'$ is strictly less than or strictly greater than zero, so small balls (with positive probability) around these states also have the same sign for this determinant. As a consequence, we have that $P(SDE) > 0$, $P(ADE) > 0$; the actual values depend on the specific measure used. The point is that, with no additional requirement on the measure, both situations, asymptotic or sudden death, can be found for this dynamics.

Since this dynamics does not have asymptotic entangled states, one has $SBE = AE = \emptyset$.

Case 2(b). More than one asymptotic states with points in the border of S with E
For more than one asymptotic state in this geometric situation we can distinguish four subcases, as discussed below.

All other points belong to Int S. Two non-interacting qubits subjected to two independent phase reservoirs provide an example. The dynamics (interaction picture implied) is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{D}_A \otimes I[\rho] + I \otimes \mathcal{D}_B[\rho], \tag{7a}$$

where

$$\mathcal{D}_i[\rho] = \gamma(\sigma_{z,i}\rho\sigma_{z,i} - \rho), \tag{7b}$$

with γ being a positive constant. This dynamics may be implemented experimentally for ions in a trap [11]. The reservoir would be given by applying z-directed magnetic fields with random and independent magnitudes on each ion [21] (the qubits encoded in the electronic spin of the ions). It is easy to show that if we write the initial state in the computational basis, the evolution will be given by exponential decays of all non-diagonal terms and all the diagonal ones will remain constant. So the set of asymptotic states will be given by the set of three real parameters (an intersection of a four-dimensional subspace of the set of Hermitian matrices with the set of states):

$$\rho_{\mathrm{st}} = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}, \tag{8}$$

with $p_i \geqslant 0$ for $i = 1, \ldots, 4$ and $\sum_{i=1}^{4} p_i = 1$.

In this case, we have that all asymptotic states are diagonal in the computational basis and again we have $\mathrm{Det}\,\rho = \mathrm{Det}\,\rho^{\Gamma}$. Two situations are possible: these determinants are zero or positive. Again, entanglement can die asymptotically or suddenly as the following initial states illustrate:

$$\rho(t=0) = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & c & 0 \\ 0 & c & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}, \tag{9}$$

with $|c| > 0$ (as a consequence, $p_2 > 0$ and $p_3 > 0$). The evolution will be given by states with the same form but with $|c(t)|$ decaying exponentially, so $d^{\Gamma}(t) = p_2 p_3 (p_1 p_4 - |c(t)|^2)$. Then, it is evident that, if p_1 or p_4 are initially zero, entanglement will decay only asymptotically to a state in the border of S. But if both of them are non-zero and $p_1 p_4 < |c(0)|^2$, then it will die suddenly while the state converges to (a state in) the interior of S. For $p_1 p_4 \geqslant |c(0)|^2$, the complete trajectory will remain in S.

Although examples of both situations can be provided, the typical case is definitely sudden death of entanglement[5]. As this dynamics does not exhibit asymptotic states with entanglement, $SBE = AE = \emptyset$.

All other points belong to E. For this case we chose a situation where both qubits are identical (but distinguishable) and interact collectively with a common reservoir, as happens with two spatially close two-level atoms in a thermal field (close compared to the wavelength defined by their transition). The dynamics of this situation can be described by the following master equation (also in the interaction picture) [22]:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \gamma(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-) + \gamma'(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+), \tag{10a}$$

with $J_{\pm} = \sigma_{\pm,A} + \sigma_{\pm,B}$. A convenient way to analyze this dynamics is to write the equations of motion for the density matrix elements in the basis composed of the states $\{|11\rangle, |\Psi_+\rangle, |00\rangle, |\Psi_-\rangle\}$, resulting in

[4] In fact, the condition $\mathrm{Det}\,\rho = 0$ implies $P(\partial D) = 0$. In this case, we also have $ADE = ADE'$ and $SDE = SDE'$. Given that the sorted state is entangled, it will exhibit SDE for sure if $\mathrm{Det}\,\rho > 0$ since it will converge to a state in $\mathrm{Int}\,S$. So, since $P(\mathrm{Int}\,D) = 1$, we have $P(SDE) = P(SDE \cap \mathrm{Int}\,D) = P(E \cap \mathrm{Int}\,D) = P(E)$, i.e. $P(SDE|E) = 1$. Equivalently, a state can exhibit ADE only if $\mathrm{Det}\,\rho = 0$ so that it will converge to a state in ∂S; hence, $P(ADE) \leqslant P(\partial D) = 0$.

$$
\begin{aligned}
\dot{\rho}_{11} &= -2\gamma\rho_{11} + 2\gamma'\rho_{22},\\
\dot{\rho}_{22} &= 2\gamma(\rho_{11} - \rho_{22}) + 2\gamma'(\rho_{33} - \rho_{22}),\\
\dot{\rho}_{33} &= 2\gamma\rho_{22} - 2\gamma'\rho_{33},\\
\dot{\rho}_{44} &= 0,\\
\dot{\rho}_{12} &= -2\gamma\rho_{12} + 2\gamma'\rho_{23} - \gamma'\rho_{12},\\
\dot{\rho}_{13} &= -\gamma\rho_{13} - \gamma'\rho_{13},\\
\dot{\rho}_{14} &= -\gamma\rho_{14},\\
\dot{\rho}_{23} &= -\gamma\rho_{23} + 2\gamma\rho_{12} - 2\gamma'\rho_{23},\\
\dot{\rho}_{24} &= -\gamma\rho_{24} - \gamma'\rho_{24},\\
\dot{\rho}_{34} &= -\gamma'\rho_{34}.
\end{aligned}
\tag{10b}
$$

The reservoir at zero temperature corresponds to the case $\gamma' = 0$. It is easy to see from the equations of motion that the complete subspace $span\{|00\rangle, |\Psi^-\rangle\}$ is stationary under this dynamics. By convexity, the stationary states have the following form:

$$
\rho_{\rm st} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\rho_{44} & \rho_{34} \\ 0 & 0 & \rho_{34}^* & \rho_{44} \end{pmatrix} \tag{11}
$$

and can be identified with a Bloch ball inside D. All states have a null determinant, so all of them are at the boundary of D. It is readily seen (representing these states in the computational basis) that $\mathrm{Det}\,\rho^\Gamma = -(\rho_{44}/2)^4$, which is null only if $\rho_{44} = 0$ and is negative otherwise, so the set does not have any points in the interior of S: this Bloch ball just touches the set of separable states at one point. Some things can be inferred immediately from the geometry of this set: (a) the entanglement of the system may never die (the singlet state is stationary, for instance); (b) it can be created: take any initially separable state ρ with non-zero population in the singlet state; (c) in principle, the entanglement can die asymptotically or suddenly. In fact, initial states leading to this situation exist but only (a) and (b) are 'typical'.

A helpful fact about this problem is that the singlet population is constant through the evolution so, if this population is positive on the initial state, it will converge to an entangled state. Since the event formed by all states with non-zero population has probability 1, we immediately infer $P(AE) = 1$, $P(SBE) = P(S)$ or $P(SBE|S) = 1$, that is, if one chooses randomly an initial state, regardless of the fact if it is entangled or not, it will evolve to an entangled state with probability 1. From this, we immediately see that $P(ADE) = P(SDE) = P(ADE') = P(SDE') = 0$, i.e. the probability of choosing an initially entangled state whose entanglement will vanish is zero.

Nevertheless, one can find atypical specific examples exhibiting *SDE* and *ADE*. Consider, for instance, the family of initial states where the only non-vanishing matrix elements (in the basis mentioned above) are $\rho_{11}, \rho_{22}, \rho_{33}$. From equations (10$b$), it follows that these will continue to be the only non-vanishing elements. If also $\rho_{11} = 0$ their behavior is quite simple: $\rho_{11}(t) = 0$, $\rho_{22}(t) = \rho_{22}\,\mathrm{e}^{-2\gamma t}$, $\rho_{33}(t) = 1 - \rho_{22}(t)$. So if $\rho_{22} \neq 0$ the state will remain entangled for all times (mixture of a Bell state with an orthogonal separable state) and will die asymptotically, i.e. exhibit *ADE*. On the other hand, if $\rho_{11} \neq 0$ the behavior of these matrix elements is still simple and the determinant of the partial transpose will acquire the following form: $\mathrm{Det}\,\rho^\Gamma(t) = \rho_{11}\,\mathrm{e}^{-2\gamma t} + P(t)\,\mathrm{e}^{-4\gamma t}$, where $P(t)$ is a second degree polynomial with

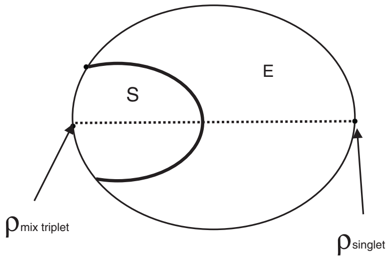

Figure 2. Set of asymptotic states for two qubits interacting with a common reservoir at infinite temperature. Here, $\rho_{\text{mix triplet}} = \frac{1}{3}(|11\rangle\langle 11| + |\Psi_+\rangle\langle\Psi_+| + |00\rangle\langle 00|)$.

coefficients determined by the initial density matrix elements. Since $\rho_{11} \neq 0$, this determinant will be positive after a certain instant of time, i.e. the state will be always separable after that instant. If, e.g., $\rho_{33} = 0$, $\rho_{11} \neq 0$, $\rho_{22} \neq 0$, the initial state is entangled and therefore will exhibit *SDE*.

Some points belong to Int S and others to E. The reservoir used in the last subcase, if taken at positive temperature, provides this example and, to simplify the problem we take the infinite temperature limit ($\gamma = \gamma'$ in equation (10*a*)). It is interesting that, irrespective of temperature, the singlet state is stationary and also the singlet population of any state (the singlet spans a one-dimensional *decoherence free subspace* for this model). From the equations of motion, it immediately follows that the stationary states are

$$\rho_{\text{st}} = \begin{pmatrix} \dfrac{1-p}{3} & 0 & 0 & 0 \\ 0 & \dfrac{1-p}{3} & 0 & 0 \\ 0 & & \dfrac{1-p}{3} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{12}$$

where p is the singlet population of the state. That is, they are the Werner states (with a different parametrization).

The determinant of the partial transpose (with respect to the computational basis, of course) is simply $(3 - 12p^2)/36$, being negative only if $p > 1/2$. The set of stationary states forms a line segment in D with both ends, those with $p = 0$ or $p = 1$, on the border of D, one of them in the interior of S (relative to D) and the other in E, respectively, and the line intersecting the border between S and D when $p = 1/2$ (see figure 2).

Since the singlet population remains fixed in the dynamics, it allows us to identify the asymptotic state of any given initial condition. An initial state will have non-zero entanglement asymptotically if, and only if, $\rho_{44} > 1/2$, so we have $P(AE) = P(D_{>1/2} = \{\rho \in D|\rho_{44} > 1/2\}) > 0$. Of course, $P(SBE) = P(D_{>1/2} \cap S)$. Since a state can exhibit *ADE* iff it relaxes to a state in the border between S and E, we have $P(ADE') \leqslant P(ADE) \leqslant P(ADE \cap \{\rho \in D|\rho_{44} = 1/2\}) = 0$. So *ADE* is atypical for this dynamics but *SDE*, on the other hand, has a non-zero probability. In fact, an initially entangled state has *SDE* iff $\rho_{44} < 1/2$, so $P(SDE') = P(E \cap \{\rho \in D|\rho_{44} < 1/2\}) > 0$.

All points belong to ∂S. The combination of two reservoirs used in former examples will provide this case. If we have qubit A subjected to spontaneous decay and B to a phase reservoir the system will have the desired behavior, a situation that may occur experimentally if we entangle an atom in vacuum with a spin subjected to a stochastic magnetic field. That is, the system dynamics would be described by a master equation of form (7*a*) (again in the interaction picture), but with $\mathcal{D}_A$ given by equation (3*b*) (with $i = A$ and $\gamma'_A = 0$) and $\mathcal{D}_B$ by equation (7*b*) (with $i = B$). It is easy to see that the set of asymptotic states will be constituted by the product states where A is in the $|0\rangle$ state and B in a state described by a diagonal matrix (in the computational basis), so the global states read as

$$\rho_{\mathrm{st}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}, \tag{13}$$

for $0 \leqslant p \leqslant 1$. Whatever the value of p we have $\mathrm{Det}\,\rho_{\mathrm{st}} = \mathrm{Det}\,\rho_{\mathrm{st}}^{\Gamma} = 0$, so they indeed belong to ∂S. Again we have $SBE = AE = \emptyset$ for this dynamics, since there are no entangled asymptotic states, but to analyze the probability of the other events we use the exact solution for the dynamics and write the determinant of the partial transpose in the form

$$\mathrm{Det}\,\rho(t)^{\Gamma} = f_1(\rho)\,\mathrm{e}^{-\lambda_1 t} + \cdots + f_n(\rho)\,\mathrm{e}^{-\lambda_n t}, \tag{14}$$

where the functions f_i depend on the initial state only, while $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. In this way, as long as $f_1 \neq 0$, the asymptotic sign of $\mathrm{Det}\,\rho(t)^{\Gamma}$ will by given by the sign of f_1. Denoting by γ_A and γ_B the decay rate for each reservoir, it so happens that $\lambda_1 = 2\gamma_A$ and $f_1 = \mathrm{Det}\,\rho'$, where

$$\rho' = \begin{pmatrix} \rho_{11} & \rho_{12}^* & 0 & 0 \\ \rho_{12} & \rho_{11}+\rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34^*} \\ 0 & 0 & \rho_{34} & \rho_{33}+\rho_{44} \end{pmatrix}. \tag{15}$$

Since this matrix is positive definite (given that ρ is), the system will reach its asymptotic state from the interior of the separables if $f_1(\rho') > 0$. But the event $f_1(\rho') = 0$ have zero probability, so we may conclude that $P(SDE) = P(SDE') = P(E)$, while $P(ADE) = P(ADE') = 0$, that is, a sorted initial entangled state will exhibit sudden death of entanglement with certainty, in contrast to case 2(a) where both sudden and asymptotic death had positive probabilities. Still, it is possible to find specific states where asymptotic death takes place. For instance, consider the set of initial states:

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Then the partial transpose determinant will be $\mathrm{Det}\,\rho(t)^{\Gamma} = -|\rho_{23}(t)|^2\rho_{22}(t)\rho_{33}(t)$, being negative for all t if ρ_{23}, ρ_{22} and ρ_{33} are initially different from zero, so the entanglement dies asymptotically.

Case 3(a). One asymptotic state in E

The most natural way to realize a dynamics with this property is through a thermal reservoir. This time, though, interacting qubits and a common thermal reservoir are needed, that is, a reservoir that takes any initial state to the Gibbs state $Z^{-1}\exp(-\beta H)$, with $Z = \mathrm{Tr}\exp(-\beta H)$ and H stands for the Hamiltonian describing the closed dynamics of the qubits. Typically, the ground state of interacting qubits Hamiltonian is non-degenerate and entangled, so if β is large enough we obtain the desired dynamics.

Dynamics with these asymptotic states can be engineered using Lindblad autonomous equations, at least formally. Actually, by fixing an arbitrary state for the system, we find many Linbdbladians that have this state as the only asymptotic state, in particular there are those with only one Lindblad operator and null Hamiltonian part [23]. The specific Lindbladian of course will depend on the specific interaction between qubits and reservoir (if the dynamics could be described by a Lindblad equation in the first place).

To give a more specific picture, consider, for instance, two interacting qubits described by the following Hamiltonian:

$$H = \tfrac{1}{2}\omega\sigma_{z,A} + \tfrac{1}{2}\omega\sigma_{z,B} + g(\sigma_{+,A}\sigma_{-,B} + \sigma_{-,A}\sigma_{+,B}) \tag{17}$$

with ω, g being positive constants satisfying $g > \omega$ (i.e. strong coupling limit). The eigenvalues for this Hamiltonian are, in crescent order, $-g, -\omega, \omega, g$, with respective eigenvectors $|\Psi_-\rangle, |00\rangle, |11\rangle, |\Psi_+\rangle$, leading to an entangled ground state. Denote by $|i\rangle, i = 1, \ldots, 4$, these eigenvectors according to their eigenvalues order. We may consider a thermal reservoir at null temperature that induces decays between any two of these states in a Markovian way, such that the dissipator would be

$$\mathcal{D}[\rho] = \sum_{i<j}\gamma_{ij}(2\sigma_{ij}\rho\sigma_{ji} - \sigma_{jj}\rho - \rho\sigma_{jj}), \tag{18}$$

where $\sigma_{ij} = |i\rangle\langle j|$ and γ_{ij} are non-negative constants. A dissipator of this type can be derived from a microscopic model, for instance adapting the results of [24] to the Hamiltonian considered here.

As in case 1(a), the events and probabilities we are interested in are trivial: $SBE = S$, $AE = D$, i.e. every initial state will acquire entanglement for large times, in particular the separable ones, so $P(SBE) = P(S)$ and $P(AE) = 1$. Since the entanglement never vanishes, $ADE = ADE' = SDE = SDE' = \emptyset$.

Case 3(b). Several asymptotic states in E

Examples for this case can be provided just by the same trick used in case 1(b): we take any Lindbladian with only one asymptotic entangled state and insert a time-varying coupling which multiplies the dissipator. The same reasoning can be applied with respect to the asymptotic states for the subsequent dynamics (in the interaction picture), so, if the decay rate of the coupling is small enough, the set of asymptotic states will be constituted by a small 'blurring' around the asymptotic state of the dynamics with constant coupling.

In contrast to case 1(b), though, in which entanglement is concerned, it is important now whether the dynamics is given in the Schrödinger or interaction pictures, because their correspondence is given by global unitary transformations. For the same reason as before, the dynamics will not be relaxing in the Schrödinger picture, but one can still find a non-trivial asymptotic set, this time entirely contained in *E*. Indeed, diminishing the decay rate of the reservoir couplings, we can diminish at will the diameter of the set of stationary states in the interaction picture which, in its turn, always contain the Gibbs state of the system. Now, unitary transformations are isometries for practically all relevant metrics, so the set of asymptotic states in the interaction picture is mapped to sets with the same diameter in the Schrödinger picture. But these unitary transformations have the Gibbs state as a fixed point; hence, these sets always contain it. Since *E* is open, given that their diameter is small enough, we can be sure that they always fall entirely inside of it.

As a consequence of the discussion in the above paragraph, the events and probabilities we are considering in this paper are identical to those in the last example.

4. Conclusions

In this paper, we review the classification of the possible dynamics of entanglement based on the relative geometry of the sets of asymptotic and separable states. We provided examples for all possible classes, including the previously unknown cases with more than one asymptotic state, but avoiding the boundary ∂S. In giving these examples, it was sufficient to use two-qubit dynamics dictated by equations of motion in the Lindblad form (including non-autonomous dynamics exactly for those previously hard examples). In each case, the existence of sudden death of entanglement, asymptotic death of entanglement, sudden birth of entanglement and asymptotic entanglement were analyzed from a more precise point of view, looking at the *probabilities* that each of these phenomena occurs if one chooses a random initial state and a suitable probability measure on the set of quantum states.

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